A note on the minimum size of an orthogonal array

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Abstract

It is an elementary fact that the size of an orthogonal array of strength t on k factors must be a multiple of a certain number, say L_t, that depends on the orders of the factors. Thus L_t is a lower bound on the size of arrays of strength t on those factors, and is no larger than L_k, the size of the complete factorial design. We investigate the relationship between the numbers L_t, and two questions in particular: For what t is $L_t < L_k$? And when $L_t = L_k$, is the complete factorial design the only array of that size and strength t? Arrays are assumed to be mixed-level.

We refer to an array of size less than L_k as a *proper fraction*. Guided by our main result, we construct a variety of mixed-level proper fractions of strength $k-1$ that also satisfy a certain group-theoretic condition.

Key words. Conjugacy; factorial design; mixed-level array; multiset; orthogonal array; strength
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1 Introduction

Let D be an orthogonal array of size N on k factors, the ith factor having s_i values or levels. It is easy to see that if t is the strength of D, then N is a multiple of

$$L_t = \operatorname{lcm}\left(\prod_{i \in I} s_i, |I| = t\right), \tag{1}$$

where $|I|$ is the number of indices in I. These points are reviewed in detail below.

Obviously L_t is a lower bound for N, although an array of strength t and size $N = L_t$ may not exist. Moreover, $L_t \leq s_1 \cdots s_k = L_k$, the size of the complete factorial design on

these factors, since the complete design certainly has strength t. From the point of view of applications, it is of interest to know when $L_t < L_k$, since in this case one may seek a design that is a proper fraction of the full factorial design. Theorem 2.1 gives a simple criterion based on the orders s_i to determine those t for which we have $L_t < L_k$.

1.1 Definitions and notation

As indicated above, we use $|I|$ to denote the cardinality of the set I.

A *complete factorial design* is a Cartesian product $A_1 \times \cdots \times A_k$, where A_i is a finite set. In statistical design, A_i is the set of *levels* of factor i, and the elements of $A_1 \times \cdots \times A_k$ are *treatment combinations*. We will refer to $s_i = |A_i|$ as the *order* of factor i. An *orthogonal array* or *design* D on these factors is a multisubset (a subset with possible repetitions) of $A_1 \times \cdots \times A_k$. The *size* of the array, N, is the number of elements (or *runs*), counting multiplicities. The design is *symmetric* if $s_1 = \cdots = s_k$, and otherwise is asymmetric or *mixed-level*. We will say that a design is a *proper fraction* if it is a a proper subset of a complete factorial design.

If we write the elements of a design D as columns, then we may represent D as a $k \times N$ matrix of symbols. The *projection* of D on j factors is then the $j \times N$ submatrix consisting of the rows corresponding to those factors. We say that D has *strength* t if for any subset $I \subset \{1, \ldots, k\}$ of size t, the projection of D on the factors indexed by I consists of λ_I copies of the Cartesian product $\prod_{i \in I} A_i$, for some λ_I. Evidently we have

$$N = \lambda_I \prod_{i \in I} s_i,$$

from which it follows easily that N is a multiple of the number L_t given by (1). In particular, $L_1 = \mathrm{lcm}(s_1, \ldots, s_k)$ and $L_k = s_1 \cdots s_k$, and it is easy to see that

$$L_t | L_{t+1}$$

for each $t = 1, \ldots, k-1$ (each product in L_t is contained in a product in L_{t+1}). Therefore

$$L_1 \leq L_2 \leq \cdots \leq L_k. \tag{2}$$

In a symmetric design with $s_1 = \cdots = s_k = s$ we have $L_t = s^t$.

For each t, L_t is the smallest possible size of an array of strength t. An array of this size may not actually exist. For example, in a symmetric design on $k = 4$ factors, each with 6 levels, we have $L_2 = 36$, but there is no array of strength 2 and size 36, since this would be equivalent to two mutually orthogonal Latin squares of order 6 (see, e.g., [3], pages 11 and 33).

As indicated above, it is of interest to know for which t we have $L_t < L_k$. Theorem 2.1 gives a useful criterion, and also strengths the inequality (2).

2 The numbers L_t

For each $I \subset \{1, \ldots, k\}$ let

$$e_I = \gcd(s_i, i \in I),$$

and consider those I for which $e_I > 1$. Let

$$d = \max\{|I| : e_I > 1\}.$$

Thus $|I| > d$ implies $e_I = 1$. We see that $1 \le d \le k$, and $d = 1$ iff the orders s_i are pairwise relatively prime. At the other extreme, $d = k$ iff $\gcd(s_1, \dots, s_k) > 1$. For example, $d = k$ if the design is symmetric.

In the proof below we need to choose a subset I of size d for which $e_I > 1$. There can be more than one such subset: for example, if the numbers s_i are $8, 12, 18, 27$, then $d = 3$, since the four numbers have no common factor but $\{8, 12, 18\}$ and $\{12, 18, 27\}$ have gcd greater than 1; the corresponding sets $I \subset \{1, \dots, 4\}$ are $\{1, 2, 3\}$ and $\{2, 3, 4\}$.

Theorem 2.1. *We have $L_1 < \cdots < L_d = \cdots = L_k$. In particular, $L_t = L_k$ iff $t \ge d$.*

Before proving this theorem, we mention two useful facts. First, if $S_1, \dots, S_n$ are sets of positive integers and $S = \cup_i S_i$, then

$$\operatorname{lcm}(S) = \operatorname{lcm}(\operatorname{lcm}(S_1), \dots, \operatorname{lcm}(S_n)). \tag{3}$$

Second, for positive integers $a_1, \dots, a_n$ we have

$$\operatorname{lcm}\left(\prod_{i \in I} a_1,\ |I| = n - 1\right) = \frac{a_1 \cdots a_n}{\gcd(a_1, \dots, a_n)}. \tag{4}$$

These can both be proved prime-by-prime: Define $\operatorname{ord}_p(b)$, the *order* of the prime p in b, to be the largest power of p dividing b. To prove (4), for example, let $f_i = \operatorname{ord}_p(a_i)$. Then (4) is the statement that for each prime p,

$$\max_j \left(\sum_{i=1}^n f_i - f_j\right) = \sum_{i=1}^n f_i - \min_j f_j.$$

(Property (4) is familiar in the case $n = 2$, namely, $ab = \operatorname{lcm}(a, b) \gcd(a, b)$.)

Proof of Theorem 2.1. Fix t. In order to compare L_t and L_{t+1}, we begin by organizing the subsets $I \subset \{1, \dots, k\}$ of size t into overlapping families B_J indexed by the subsets J of size $t + 1$. Namely, we put

$$B_J = \{I \subset J\colon |I| = t\}.$$

Let $B = \{I \subset \{1, \dots, k\}\colon |I| = t\}$. Then

$$B = \bigcup_{|J|=t+1} B_J.$$

Now from (3) and (4) we have

$$
\begin{aligned}
L_t &= \operatorname{lcm}\left(\prod_{i\in I} s_i, |I| = t\right) \\
&= \operatorname{lcm}\left(\operatorname{lcm}\left(\prod_{i\in I} s_i, I \in B_J\right), |J| = t+1\right) \\
&= \operatorname{lcm}\left(\prod_{i\in J} s_i/e_J, |J| = t+1\right). \qquad (5)
\end{aligned}
$$

If $t \geq d$ then the expression (5) is exactly L_{t+1}. For if $t \geq d$ then $e_J = 1$ whenever $|J| = t+1$. Thus $t \geq d$ implies that $L_t = L_{t+1}$.

If $t < d$, we claim that (5) is strictly less than L_{t+1}. To see this, fix a prime p that divides exactly d of the numbers $s_1, \ldots, s_k$. (This is possible as there exists a set $I \subset \{1, \ldots, k\}$ with $|I| = d$ and $e_I > 1$; let $p|e_I$.) Our claim follows as soon as we show that the order of p in (5) is strictly less than $\operatorname{ord}_p(L_{t+1})$. To do this, we need to show that this holds for each term in (5), that is,

$$
\operatorname{ord}_p\left(\prod_{i\in J} s_i/e_J\right) < \operatorname{ord}_p(L_{t+1}) \qquad (6)
$$

for each $J \subset \{1, \ldots, k\}$ with $|J| = t+1$.

Let $f_i = \operatorname{ord}_p(s_i)$. Note that exactly d of the integers f_i are positive. Renumbering if necessary, we may assume without loss of generality that

$$
f_1 \geq f_2 \geq \cdots \geq f_d > 0 = f_{d+1} = \cdots = f_k.
$$

Now on the one hand, the order of p in L_{t+1} is the maximum of the orders of p in the products $\prod_{i\in J} s_i$, $|J| = t+1$. But this maximum is attained for $J = \{1, \ldots, t+1\}$, and equals $f_1 + \cdots + f_{t+1}$; that is,

$$
\operatorname{ord}_p(L_{t+1}) = \sum_{i=1}^{t+1} f_i. \qquad (7)
$$

On the other hand, for any $J \subset \{1, \ldots, k\}$ we have

$$
\operatorname{ord}_p(e_J) = \min_{i\in J} f_i,
$$

and so

$$
\operatorname{ord}_p\left(\prod_{i\in J} s_i/e_J\right) = \sum_{i\in J} f_i - \min_{i\in J} f_i.
$$

We focus on sets J with $|J| = t+1$, and consider two cases:

- $\min_{i\in J} f_i > 0$: Here

$$\sum_{i\in J} f_i - \min_{i\in J} f_i < \sum_{i\in J} f_i \le \sum_{i=1}^{t+1} f_i.$$

- $\min_{i\in J} f_i = 0$: In this case, $f_i = 0$ for at least one $i \in J$, so

$$\sum_{i\in J} f_i - \min_{i\in J} f_i = \sum_{i\in J} f_i < \sum_{i=1}^{t+1} f_i.$$

In either case (6) holds, and therefore the order of p in (5) is less than (7), which is what we needed to prove. $\square$

Theorem 2.1 provides useful information about possible orthogonal arrays without our directly having to calculate the numbers L_t. In particular, it gives a necessary condition for the existence of a proper fraction of given strength, as illustrated by the examples below. A special case of the theorem is given in [2].

Example 1. In a 2^i3^j experiment, that is, one having $i+j$ factors of which i have 2 levels and j have 3, we see that $d = \max(i, j)$. In this case, no proper fraction has strength $t \ge \max(i, j)$.

Example 2. Consider a $2 \times 3 \times 5 \times 6 \times 10 \times 15$ experiment. Any subset of $\{2, 3, 5, 6, 10, 15\}$ of size 4 must have $e_I = 1$, as two of its elements must be relatively prime. On the other hand, there are subsets of size 3 that have $e_I > 1$ – for example, $\{2, 6, 10\}$ (that is, $I = \{1, 4, 5\}$). Thus $d = 3$, and so no proper fraction has strength 3.

Example 3. In order for there to exist a proper fraction of strength $k - 1$ in an experiment with k factors, it is necessary that $d = k$. That is, the orders $s_1, \ldots, s_k$ must share a common factor. In Section 4 we give methods of construction that generate a number of proper fractions of strength $k - 1$ that also satisfy a group-theoretic condition when the set of treatment combinations is a nonabelian group. The condition and its origin are discussed there.

3 Uniqueness

When $L_t = L_k$, the smallest array of strength t has the same size as the complete factorial design, and it is natural to ask whether the complete factorial is the only array of its size of strength t. It is easy to see that for strength $t = 1$ this is not true. For suppose factor i is indexed by the set A_i where $|A_i| = s_i$. Fill a $k \times L_k$ matrix by putting L_k/s_i copies of A_i in row i, in an arbitrary order. The resulting array has strength $t = 1$ and size L_k, but it is easy to fill the rows in such a way that the columns do not consist of all the elements of $A_1 \times \cdots \times A_k$.

Of course, the rows of an array of strength 1 can be filled independently of each other, and so one might conjecture that such a construction is impossible if we require strength 2 or higher, but in fact this is not true either. Consider a $3 \times 2 \times 2$ factorial experiment and let $A_1 = \{0, 1, 2\}$ and $A_2 = A_3 = \{0, 1\}$. In this case $L_2 = 12$, the size of the complete factorial

design, and so there is no smaller orthogonal array of size 12 and strength 2 on these factors. We easily check that the 3×12 array

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

has strength 2, but it is clearly not the complete design, as some ordered triples occur more than once while others are missing. It would be interesting to know whether there are applications in which such an array might be a useful alternative to the full factorial design.

4 Constructions

In this section we construct a number of mixed-level orthogonal arrays of strength $k-1$ on k factors. They are all proper fractions and in addition satisfy a group-theoretic property (a "conjugacy" condition) given in [1] that we describe below. As we saw in Example 3, Theorem 2.1 requires that the orders s_i share a common factor.

Our examples consist of $k = 3$ to 6 factors, with orders $s_i \leq 10$. Verification that each array satisfies our two conditions (strength $k-1$ and conjugacy) was done by computer, for which the code can be found in [2]. Only those cases listed in Table 1 (below) have been checked.

4.1 Conjugacy

Throughout this section we will suppose that the sets A_i indexing the levels of the factors are groups, and will use G_i rather than A_i as a more suggestive notation. The set $G = G_1 \times \cdots \times G_k$ of treatment combinations is a group, and is abelian iff all the factors G_i are abelian.

Recall that the elements a and b of G are conjugate if $b = gag^{-1}$ for some $g \in G$. Conjugacy is an equivalence relation on G, and so partitions G into *conjugacy classes*. In an abelian group, the conjugacy classes are all singleton sets. It is an elementary fact that the conjugacy classes of $G_1 \times \cdots \times G_k$ are of the form $C_1 \times \cdots \times C_k$ where C_i is a conjugacy class of G_i. The condition we require is that *the design D be a union of conjugacy classes of G*.[1]

In all our examples G_1, and therefore G, is a nonabelian group, while G_i will be abelian for $i > 1$. Thus the conjugacy classes of G are essentially those of G_1 – namely, of the form $C \times \{g_2\} \times \cdots \times \{g_k\}$ where C is a conjugacy class of G_1 and $g_i \in G_i$. The nonabelian groups we will use are the following, where e will denote the identity element.

- S_3, the symmetric group on 3 letters. We have $S_3 = \{e, x, y, a, b, c\}$, where x and y are the 3-cycles and a, b and c are the 2-cycles. We will use either of two orderings:

    1. $e \mid x\ y \mid a\ b\ c$.

[1]The condition assumed in [1] is that the counting function of the multiset D be constant on conjugacy classes. Since we are constructing proper fractions, D is a set and its counting function is its ordinary indicator function, which is constant on conjugacy classes iff D is a union of such classes.

2. $e \mid a\ b\ c \mid x\ y$.

The vertical lines are inserted merely to indicate the conjugacy classes.

- $Dih_4 = \{e, q, r, s, a, b, x, y\}$, the symmetries of the square, where q is the half-turn, r and s are the quarter-turns, a and b are the reflections about the diagonals, and x and y are the reflections about the lines joining the midpoints of the opposite sides. The conjugacy classes are $e \mid q \mid r\ s \mid a\ b \mid x\ y$.

- $Dih_5 = \{e, a, b, c, d, v, w, x, y, z\}$, the symmetries of the regular pentagon, where a and d are the rotations of $\pm 72°$, b and c are the rotations of $\pm 144°$, and v, w, x, y and z are the reflections about a line from a vertex to the midpoint of the opposite side. The conjugacy classes are $e \mid a\ d \mid b\ c \mid v\ w\ x\ y\ z$.

In each of the above, the ordering of conjugacy classes is fixed, and the ordering of elements within each class is fixed but arbitrary. These fixed orderings are assumed in the constructions below. In general, the group Dih_n is a *dihedral group*, and represents the symmetries of the regular n-gon. Of course, S_3 is also Dih_3.

For abelian groups we will use $\mathbb{Z}_n$, the integers modulo n, whose elements are $0, 1, \ldots, n$. We will fix this order.

4.2 Construction

If $s_1 = 6, 8$ or 10, we let $G_1 = S_3, Dih_4$ or Dih_5, respectively. For $i \geq 2$, we let $G_i = \mathbb{Z}_{s_i}$. Our construction has the following cases.

gcd($s_1, \ldots, s_k$) = 2 or 4: We construct an array of minimal size L_{k-1}. Defining the integers

$$v_1 = \frac{L_{k-1}}{s_1},$$
$$v_j = \frac{L_{k-1}}{s_2 \cdots s_j} \text{ for } j \geq 2,$$

we fill in the rows of the array as follows:

Row 1: Write the elements of G_1 in the fixed order, the whole sequence repeated v_1 times. If $s_1 = 6$, use S_3 in its first ordering.

Row 2: Write the elements of $G_2 (= \mathbb{Z}_{s_2})$ in the fixed order, repeating each element v_2 times.

Rows 3 through $k-1$: Write the elements of $G_j (= \mathbb{Z}_{s_j})$ in the fixed order, repeating each element v_j times. Repeat the whole pattern until the row is complete (a total of L_{k-1} entries).

Row k: Write the elements of G_k, repeating each element v_k times, in the following pattern: first in the given order, then in the reverse order, alternating in that way until the row is complete.

Example 4. The following arrays illustrate the construction method we have described.

- 1/2 fraction of a $6 \times 2 \times 2 \times 2$ design, strength 3. Here $G = S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\begin{pmatrix} e & x & y & a & b & c & e & x & y & a & b & c & e & x & y & a & b & c & e & x & y & a & b & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- A 1/2 fraction of an $8 \times 2 \times 2$ design, strength 2. $G = Dih_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\begin{pmatrix} e & q & r & s & a & b & x & y & e & q & r & s & a & b & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- A 1/2 fraction of a $10 \times 2 \times 2$ design, strength 2. $G = Dih_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{pmatrix} e & a & b & c & d & v & w & x & y & z & e & a & b & c & d & v & w & x & y & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- A 1/4 fraction of an $8 \times 4 \times 4$ design, strength 2. $G = Dih_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Vertical lines are inserted to reveal the pattern in the last row.

$$\left(\begin{array}{cccccccc|cccccccc|cccccccc|cccccccc} e & q & r & s & a & b & x & y & e & q & r & s & a & b & x & y & e & q & r & s & a & b & x & y & e & q & r & s & a & b & x & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 0 & 0 \end{array}\right)$$

$\mathbf{gcd}(s_1, \ldots, s_k) = \mathbf{6}$: We in fact consider only symmetric "6^k" designs, so that $L_{k-1} = 6^{k-1}$. The method described above may be used to create arrays of this minimum size (and therefore 1/6 fractions), but the arrays will not satisfy the conjugacy requirement. If we modify the integers v_i as follows:

$$v_1 = \frac{3L_{k-1}}{s_1} = 3 \cdot 6^{k-2},$$
$$v_j = \frac{3L_{k-1}}{s_2 \cdots s_j} = 3 \cdot 6^{k-j} \text{ for } j \geq 2,$$

then the method will produce 1/2 fractions of strength $k-1$. Note that the first $v_k = 3$ elements of $G_1 = S_3$ is a union of conjugacy classes of S_3.

We don't present an example since $N = (1/2)6^k$ rather large.

$\mathbf{gcd}(s_1, \ldots, s_k) = \mathbf{3}$: We apply this to $6 \times 3 \times 3 \cdots$ factorial experiments. Here we alter both the integers v_i and the steps of construction, since the original steps will produce arrays

of minimum size (1/3 fractions) that do not satisfy the conjugacy property. We set

$$\begin{aligned} v_1 &= \frac{2L_{k-1}}{s_1} = 2 \cdot 3^{k-2}, \\ v_j &= \frac{2L_{k-1}}{s_2 \cdots s_j} = 4 \cdot 3^{k-j} \text{ for } j \geq 2, \end{aligned}$$

and fill in the rows of the array as follows:

Row 1: We use the *second* order of S_3. Write the elements of $G_1 = S_3$ in this fixed order, then in the reverse order, alternating until there are $N = 2L_{k-1}$.

Rows 2 through $k-1$: These steps are identical to those given above.

Row k: Write the elements of $G_k = \mathbb{Z}_3$, repeating each element $v_k = 4$ times. Then do the same, but permuting the elements of $\mathbb{Z}_3$ cyclically, and again with another cyclic permutation, continuing in this pattern until the row is filled.

This method produces 2/3 fractions.

Example 5. A 2/3 fraction of a $6 \times 3 \times 3$ design, strength 2. Here $G = S_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Vertical lines are inserted to reveal the pattern in the last row.

$$\left(\begin{array}{cccccccccccc|cccccccccccc|cccccccccccc} e & a & b & c & x & y & y & x & c & b & a & e & e & a & b & c & x & y & y & x & c & b & a & e & e & a & b & c & x & y & y & x & c & b & a & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \end{array}\right)$$

Table 1 summarizes 31 arrays constructed using the methods we have described.

References

[1] Jay H. Beder and Jesse S. Beder. Generalized wordlength patterns and strength. *Journal of Statistical Planning and Inference*, 144:41–46, 2014.

[2] Margaret Ann McComack. Constructing orthogonal arrays on non-abelian groups. Master's thesis, University of Wisconsin – Milwaukee, 2013.

[3] Damaraju Raghavarao. *Constructions and Combinatorial Problems in Design of Experiments*. Dover, 1988. Reprint of the original Wiley edition, 1971.

Table 1: This table lists arrays constructed using the methods in this paper. Strength is $k-1$ where k is the number of factors. All arrays are if minimal size L_{k-1} unless otherwise indicated. Examples are separated into four groups according to $\gcd(s_1, \ldots, s_k)$.

Complete Design	Size of Complete Design	Size of Array		Fraction
$6 \times 2 \times 2^*$	24	12		1/2
$6 \times 2 \times 2 \times 2^*$	48	24		1/2
$6 \times 4 \times 4$	96	48		1/2
$6 \times 4 \times 4 \times 4$	384	192		1/2
$6 \times 4 \times 2$	48	24		1/2
$6 \times 6 \times 2$	72	36		1/2
$6 \times 6 \times 4$	144	72		1/2
$8 \times 2 \times 2$	32	16		1/2
$8 \times 2 \times 2 \times 2$	64	32		1/2
$8 \times 2 \times 2 \times 2 \times 2$	128	64		1/2
$8 \times 2 \times 2 \times 2 \times 2 \times 2$	256	128		1/2
$8 \times 6 \times 6$	288	144		1/2
$8 \times 6 \times 6 \times 6$	1,728	864		1/2
$8 \times 4 \times 2$	64	32		1/2
$8 \times 6 \times 2$	96	48		1/2
$8 \times 6 \times 4$	192	96		1/2
$10 \times 2 \times 2$	40	20		1/2
$10 \times 2 \times 2 \times 2$	80	40		1/2
$10 \times 4 \times 4$	160	80		1/2
$10 \times 4 \times 4 \times 4$	640	320		1/2
$10 \times 6 \times 6$	360	180		1/2
$10 \times 6 \times 6 \times 6$	2,160	1,080		1/2
$10 \times 4 \times 2$	80	40		1/2
$10 \times 6 \times 2$	120	60		1/2
$10 \times 6 \times 4$	240	120		1/2
$8 \times 4 \times 4$	128	32		1/4
$8 \times 4 \times 4 \times 4$	512	128		1/4
$6 \times 6 \times 6$	216	108	$= 3L_{k-1}$	1/2
$6 \times 6 \times 6 \times 6$	1,296	648	$= 3L_{k-1}$	1/2
$6 \times 3 \times 3$	54	36	$= 2L_{k-1}$	2/3
$6 \times 3 \times 3 \times 3$	162	108	$= 2L_{k-1}$	2/3

* Another example is constructed in [1].